A Physical Approach to Large-scale Structure and Galaxy Formation

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Rossi Lectures 2007

1. The Basic Framework for Galaxy Formation
2. The Physics of Large-scale Structure and Galaxy Formation
3. Making Real Galaxies

More pedagogical lectures:

a. The Basic Structure of the Cosmological Models
b. Perturbations - the Underlying Physics of the Origin of Structure
I am just finishing writing the last chapter of the new edition of my book *Galaxy Formation*. It will be sent to Springer-Verlag this week.

The emphasis is upon the basic physics involved in astrophysical cosmology, trying to keep it as simple, but rigorous, as possible.

My objective in the Rossi lectures (and the pedagogical lectures) is to bring the story of astrophysical cosmology up to date and to show you some simple ways of understanding the underlying physics.
The Basic Framework for Galaxy and Large-scale Structure Formation

In the first lecture, I will examine the fundamentals of the cosmological models used in modern cosmology. These turn out to be remarkably successful, but we need to ask how secure these foundations are. We need to examine:

- Basic observations on which the models are based.
- Basic assumptions made in the construction of cosmological models.
- How secure is the General Theory of Relativity?
- The Cosmological Tests - supernovae Type 1a, WMAP, etc.
- The successes and challenges
The starting points for cosmological studies nowadays are the observations of the Cosmic Microwave Background Radiation by the COBE satellite in the early 1990s.

- The spectrum is very precisely that of a perfect black-body at a radiation temperature of 2.726 K.
- A perfect dipole component is detected, corresponding to the motion of the Earth through the frame in which the radiation would be perfectly isotropic.
- Away from the Galactic plane, the radiation is isotropic to better than one part in $10^5$. At this level, significant temperature fluctuations $\Delta T/T \approx 10^{-5}$ were detected on scales $\theta \geq 10^\circ$. 
WMAP Observations of the Cosmic Microwave Background Radiation (2003)

Cosmic Background Explorer (COBE)

The same features are present in the WMAP image of the sky. The WMAP experiment had much higher angular resolution than COBE.

Wilkinson Microwave Anisotropy Probe (WMAP). Galactic emission has been subtracted.
The COBE and WMAP observations have established the isotropy of the Universe, but we also need to know about its homogeneity. This has been established by large surveys of galaxies, starting with the local distribution determined by Geller and Huchra (top) and proceeding to the largest scales accessible at the present epoch by the 2dF (bottom) and SDSS surveys which each contain over 200,000 galaxies.
The Homogeneity of the Universe

The Sloan Digital Sky Survey

The large scale distribution of galaxies is clearly irregular with giant walls and holes on scales much greater than those of clusters of galaxies.

The distributions display, however, the same degree of inhomogeneity as we observe to larger and large distances in the Universe. This is quantified by the two-point correlation functions for galaxies to different distances,

\[ n(r) = n_0[1 + \xi(r)] . \quad (1) \]
The second result we need is the redshift-distance relation for galaxies – often called the Hubble diagram.

A modern version of Hubble’s law for the brightest galaxies in rich clusters of galaxies,

\[ v = H_0 r. \]

All classes of galaxy follow the same Hubble’s law. \( H_0 \) is Hubble’s constant.

This means that the Universe as a whole is expanding uniformly. Run simulation.
Basic Assumptions

The standard models are based upon two assumptions and a theory of gravity:

- The *Cosmological Principle* – we are not located at any special place in the Universe. Combined with the observations that the Universe is isotropic, homogeneous and uniformly expanding on a large scale, this leads to the Robertson–Walker metric. We only need special relativity plus the postulates of isotropy and homogeneity to derive this metric (see pedagogical lecture 1).

- *Weyl’s Postulate* is the statement that the world lines of particles meet at a singular point in the finite or infinite past. This solves the clock synchronisation problem and means that there is a unique world line passing through every point in space-time. The fluid moves along streamlines in the expansion and behaves like a perfect fluid with energy–momentum tensor $T^{\alpha\beta}$.

- *General Relativity*, which enables us to relate the energy–momentum tensor to the geometrical properties of space-time.
The Quantities which Appear in the Robertson-Walker Metric

- A *Fundamental Observer* moves in such a way that the Universe always appears to be isotropic. *Cosmic time* is time measured on the clock of a fundamental observer.
- $r$ is the *comoving radial distance coordinate* which is fixed to a galaxy for all time and which is the proper distance the galaxy would have if its world line were projected forward to the present epoch $t_0$ and its distance measured at that time.
- Note that we need other ‘distances’ to relate observables to intrinsic properties.
The Robertson-Walker Metric

The *Robertson-Walker metric* can be written in the following form:

\[
\text{d} s^2 = \text{d} t^2 - \frac{a^2(t)}{c^2}[\text{d} r^2 + \mathcal{R}^2 \sin^2(r/\mathcal{R})(\text{d} \theta^2 + \sin^2 \theta \, \text{d} \phi^2)] .
\] (2)

The metric contains one unknown function \(a(t)\), the *scale factor*, and the constant \(\mathcal{R}\) which is the radius of curvature of the geometry of the Universe at the present epoch.

- \(t\) is cosmic time as measured by a clock carried by a fundamental observer;
- \(r\) is the *comoving radial distance coordinate* which is fixed to a galaxy for all time.
- \(a(t) \, \text{d} r\) is the element of proper (or geodesic) distance in the radial direction at the epoch \(t\);
- \(a(t)[\mathcal{R} \sin(r/\mathcal{R})] \, \text{d} \theta\) is the element of proper distance perpendicular to the radial direction subtended by the angle \(\text{d} \theta\) at the origin;
- Similarly, \(a(t)[\mathcal{R} \sin(r/\mathcal{R})] \sin \theta \, \text{d} \phi\) is the element of proper distance in the \(\phi\)-direction.
Key Results from the R-W Metric

- All the physics of the expansion of the Universe is built into the function \( a(t) \), the scale factor. \( a(t) \) is normalised to the value 1 at the present epoch \( t = t_0 \).

- The radius of curvature of space \( \mathcal{R} \) changes with scale factor as \( \mathcal{R}(t) = \mathcal{R}a \). The curvature \( \kappa = 1/\mathcal{R}^2 \) and can be positive, negative or zero.

- By redshift, we mean the shift of spectral lines to longer wavelength because of their recession velocities from our Galaxy. If \( \lambda_e \) is the wavelength of the line as emitted and \( \lambda_0 \) the observed wavelength, the redshift \( z \) is defined to be

\[
  z = \frac{\lambda_0 - \lambda_e}{\lambda_e}.
\]  

- It follows directly from the R-W metric that the redshift is directly related to the scale-factor \( a \) through the relation

\[
a(t) = \frac{1}{1 + z}.
\]

This is the real meaning of redshift in cosmology - it has nothing to do with velocities!
The Time Dilation Test

A key test of the Robertson-Walker metric is that the same formula which describes the redshift of spectral lines should also apply to time intervals in the emitted and received reference frames. This has been possible through the use of Type 1a supernovae which have remarkably similar light curves (upper panel).

The lower panel shows the width of the light curves of Type 1a supernovae as a function of redshift. (a) A clear time dilation effect is observed exactly proportional to \((1 + z)\), as predicted by the Robertson-Walker metric. (b) The width of the light curves divided by \((1 + z)\).
Testing General Relativity

Traditionally, there are four tests of General Relativity

- The **Gravitational Redshift** of electromagnetic waves in a gravitational field. Hydrogen masers in rocket payloads confirm the prediction at the level of about 5 parts in $10^5$.

- The **Advance of the Perihelion of Mercury**. Continued observations of Mercury by radar ranging have established the advance of the perihelion of its orbit to about 0.1% precision with the result $\dot{\omega} = 42.98(1 \pm 0.001)$ arcsec per century. General Relativity predicts a value of $\dot{\omega} = 42.98$ arcsec per century.

- The **Gravitational Deflection of Light by the Sun** has been measured by VLBI and the values found are $(1 + \gamma)/2 = 0.99992 \pm 0.00023$.

- The **Time Delay of Electromagnetic Waves** propagating through a varying gravitational potential. While en route to Saturn, the Cassini spacecraft found a time-delay corresponding to $(\gamma - 1) = (2.1 \pm 2.3) \times 10^{-5}$. Hence the coefficient $\frac{1}{2}(1 + \gamma)$ must be within at most 0.0012 percent of unity.
Testing General Relativity

Measurements of the quantity \((1 + \gamma)/2\) from light deflection and time delay experiments. The value of \(\gamma\) according to General Relativity is unity. The time-delay measurements from the Cassini spacecraft yielded agreement with General Relativity at the level of \(10^{-3}\) percent. VLBI radio deflection measurements have reached 0.02 percent accuracy. The \textit{Hipparcos} limits were derived from precise measurements of the positions stars over the whole sky and resulted in a precision of 0.1 percent in the measurement of \(\gamma\).
Testing General Relativity – the Binary Pulsar

A schematic diagram of the orbit of the binary pulsar PSR 1913+16. The pulsar is one of a pair of neutron stars in binary orbits about their common centre of mass. There is displacement between the axis of the magnetic dipole and the rotation axis of the neutron star. The radio pulses are assumed to be due to beams of radio emission from the poles of the magnetic field distribution. Many parameters of the binary orbit and the masses of the neutron stars can be measured with very high precision by accurate timing measurements.
The width of each strip reflects the observational uncertainties in the timing measurements, shown as a percentage. The inset shows the same three most accurate constraints on the full mass plane; the intersection region has been magnified 400 times in the full figure.

If General Relativity were not the correct theory of gravity, the lines would not intersect at one point.
Gravitational Radiation of the Binary Pulsar

The binary pulsar emits gravitational radiation and so leads to a speeding up of the stars in the binary orbit. The diagram shows the change of orbital phase as a function of time for the binary neutron star system PSR 1913+16 compared with the expected changes due to gravitational radiation energy loss by the binary system. These observations enable many alternative theories of gravity to be excluded.

Note also the discovery of J0737-3039 in which both neutron stars are observed as pulsars.
Variation of the Gravitational Constant with Cosmic Epoch

Various solar system, astrophysical and cosmological tests can be made to find out if the gravitational constant has varied over cosmological time-scales.

<table>
<thead>
<tr>
<th>Method</th>
<th>$(\dot{G}/G)/10^{-13}\text{ year}^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lunar laser ranging</td>
<td>$4\pm9$</td>
</tr>
<tr>
<td>Binary pulsar PSR 1913+16</td>
<td>$40\pm50$</td>
</tr>
<tr>
<td>Helioseismology</td>
<td>$0\pm16$</td>
</tr>
<tr>
<td>Big Bang nucleosynthesis</td>
<td>$0\pm4$</td>
</tr>
</tbody>
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For binary pulsar data, the bounds are dependent upon the theory of gravity in the strong-field regime and on the neutron star equation of state. Big-bang nucleosynthesis bounds assume specific form for time dependence of $G$.

There can have been little variation in the value of the gravitational constant over the last $10^{10}$ years.
There has been some debate about whether or not the fine-structure constant \( \alpha \) has changed very slightly with cosmic epoch from observations of fine-structure lines in large redshift absorption line systems in quasars. The Australian observers reported a small decrease in the value of \( \alpha \), shown by the open rectangle. The ESO observers found little evidence for changes.
Einstein’s Field Equations

Under the assumption of isotropy and homogeneity, Einstein’s field equations of General Relativity reduce to the following pair of independent equations.

\[ \ddot{a} = -\frac{4\pi G}{3}a \left( \varrho + \frac{3p}{c^2} \right) + \frac{1}{3}\Lambda a ; \quad (5) \]

\[ \dot{a}^2 = \frac{8\pi G\varrho}{3}a^2 - \frac{c^2}{R^2} + \frac{1}{3}\Lambda a^2 . \quad (6) \]

\( a \) is the scale factor, \( \varrho \) is the total inertial mass density of the matter and radiation content of the Universe and \( p \) the associated total pressure. \( R \) is the radius of curvature of the spatial geometry of the world model at the present epoch and so the term \( -c^2/R^2 \) is a constant of integration. The *cosmological constant* \( \Lambda \) has had a chequered history since it was introduced by Einstein in 1917.
The Meaning of the Term $\rho + \frac{3p}{c^2}$

Equation (6) is referred to as *Friedman’s equation* and has the form of an energy equation, the term on the left-hand side corresponding to the kinetic energy of the expanding fluid and the first term on the right-hand side to its gravitational potential energy. The First Law of Thermodynamics in its relativistic form needs to be built into this equation.

$$dU = -p \, dV.$$  \hspace{1cm} (7)

The first law can be written in such a way that it is applicable for all relativistic and non-relativistic fluids. We write the internal energy $U$ as the sum of all the terms which can contribute to the total energy of the fluid in the relativistic sense. Thus, the total internal energy consists of the fluid’s rest mass energy, its kinetic energy, its thermal energy and so on. If we write the sum of these energies as $\varepsilon_{tot} = \sum_i \varepsilon_i$, the internal energy is $\varepsilon_{tot}V = \rho c^2 V$ and so, differentiating (7) with respect to $a$, it follows that

$$\frac{d}{da} (\varepsilon_{tot}V) = \frac{d}{da} (\rho c^2 V) = -p \frac{dV}{da}.$$  \hspace{1cm} (8)
Setting $V \propto a^3$, we find

$$\frac{d\rho}{da} = -\frac{3}{a} \left( \rho + \frac{p}{c^2} \right). \quad (9)$$

Now differentiate Friedman's equation (6) with respect to time and divide through by $\dot{a}$. We find

$$\ddot{a} = 4\pi G a^2 \frac{d\rho}{da} + \frac{8\pi G \rho a}{3} + \frac{1}{3} \Lambda a . \quad (10)$$

Now, substituting the expression for $d\rho/da$ from (9), we find

$$\ddot{a} = -\frac{4\pi G}{3} a \left( \rho + \frac{3p}{c^2} \right) + \frac{1}{3} \Lambda a , \quad (11)$$

that is, we recover (5).

Thus, equation (5) has the form of a force equation, but it also incorporates the relativistic form of the First Law of Thermodynamics as well. The pressure term can be considered a 'relativistic correction' to the inertial mass density, but it is unlike normal pressure forces which depend upon the gradient of the pressure. The term $\rho + (3p/c^2)$ can be thought of as playing the role of an active gravitational mass density.
The Cosmological Constant $\Lambda$

Einstein’s introduction of the cosmological constant predated Hubble’s discovery of the expansion of the distribution of galaxies. In 1917, Einstein introduced the $\Lambda$-term in order to incorporate *Mach’s principle* into General Relativity - namely that the local inertial frame of reference should be defined relative to the distant stars. In the process, he derived the first fully self-consistent cosmological model - the static Einstein model of the Universe.

Einstein’s first field equation is (equation 5)

$$\ddot{a} = -\frac{4\pi G}{3}a\left(\rho + \frac{3p}{c^2}\right) + \frac{1}{3}\Lambda a .$$

Einstein’s model is static and so $\ddot{a} = 0$. The model is a ‘dust model’ and so the pressure is taken to be zero. Therefore,

$$\frac{4\pi G}{3}a\rho = \frac{1}{3}\Lambda a \quad \text{or} \quad \Lambda = 4\pi G\rho .$$

Einstein’s perspective was that this formula shows that there would be no solutions of his field equations unless the cosmological constant was finite. If $\Lambda$ were zero, the Universe would be empty.
Rewriting (5),

$$\ddot{a} = -\frac{4\pi G}{3}a \left( \rho + \frac{3p}{c^2} \right) + \frac{1}{3} \Lambda a ,$$

it can be seen that, even in an empty universe, $\rho = 0, p = 0$, there is a net force acting on a test particle. If $\Lambda$ is positive, the term may be thought of as the ‘repulsive force of a vacuum’, the repulsion being relative to an absolute geometrical frame of reference. There is no obvious interpretation of this term in term of classical physics. There is, however, a natural interpretation in the context of quantum field theory.

A key development has been the introduction of Higgs fields into the theory of weak interactions. These were introduced in order to eliminate singularities in the theory and to endow the $W^\pm$ and $Z^0$ bosons with masses. The Higgs fields are scalar fields, unlike the vector fields of electromagnetism or the tensor fields of General Relativity. The simplest scalar fields have negative energy equations of state $p = -\rho c^2$. There are many different types of scalar fields in string theory.
The Cosmological Constant $\Lambda$

In the modern picture of the vacuum, there are zero-point fluctuations associated with the zero point energies of all quantum fields. The stress–energy tensor of a vacuum has a negative energy equation of state, $p = -\varrho c^2$. This pressure may be thought of as a ‘tension’ rather than a pressure. When such a vacuum expands, the work done $p\,dV$ in expanding from $V$ to $V + dV$ is just $-\varrho c^2\,dV$ so that, during the expansion, the mass-energy density of the negative energy field remains constant.

We can find the same result from (9).

$$\frac{d\varrho}{da} + 3\left(\frac{\varrho + p}{c^2}\right) = 0.$$  

It can be seen that, if the vacuum energy density is to remain constant, it follows that $p_v = -\varrho_v c^2$.

We can now relate $\varrho_v$ to the value of $\Lambda$. We can now set $\Lambda = 0$ and instead include the energy and pressure of the vacuum fields into equation (5).
The Cosmological Constant $\Lambda$

$$\dot{a} = -\frac{4\pi G R}{3} \left( \rho_m + \rho_v + \frac{3p_v}{c^2} \right),$$  \hspace{1cm} (14)

where we have included the density of ordinary mass $\rho_m$ and the mass density $\rho_v$ and pressure $p_v$ of the vacuum fields. Since $p_v = -\rho_vc^2$, it follows that

$$\dot{a} = -\frac{4\pi G a}{3} \left( \rho_m - 2\rho_v \right).$$  \hspace{1cm} (15)

As the Universe expands, $\rho_m = \rho_0 / a^3$ and $\rho_v = \text{constant}$. Therefore,

$$\ddot{a} = -\frac{4\pi G \rho_0}{3a^2} + \frac{8\pi G \rho_v a}{3}.$$  \hspace{1cm} (16)

Equation (16) has precisely the same dependence upon $a$ as of the ‘cosmological term’ and so we can formally identify the cosmological constant with the vacuum mass density.

$$\Lambda = 8\pi G \rho_v.$$  \hspace{1cm} (17)
The critical world model has $\Omega_0 = 1$, $\Omega_\Lambda = 0$ and separates the open from the closed models and the collapsing models from those which expand forever for the case $\Omega_\Lambda = 0$. This model is often referred to as the Einstein–de Sitter or the critical model. The velocity of expansion tends to zero as $a$ tends to infinity. It has a particularly simple variation of $a(t)$ with cosmic epoch,

$$a = \left(\frac{t}{t_0}\right)^{2/3} \quad \kappa = 0,$$  

(18)

where the present age of the world model is $t_0 = (2/3)H_0^{-1}$. The density of the model $\varrho_0 = 3H_0^2/8\pi G$ is known as the critical density.
Density Parameters in the Matter and Vacuum Fields

At the present epoch, \( a = 1 \) and the first field equation becomes

\[
\ddot{a}(t_0) = -\frac{4\pi G\varrho_0}{3} + \frac{8\pi G\varrho_v}{3}.
\]  

(19)

It is convenient to express densities in terms of the critical density \( \varrho_c \),

\[
\varrho_c = \left(\frac{3H_0^2}{8\pi G}\right) = 1.88 \times 10^{-26} h^2 \text{ kg m}^{-3}.
\]  

(20)

Then, the actual density of the model \( \varrho_0 \) at the present epoch can be referred to this value through a density parameter \( \Omega_0 = \varrho_0/\varrho_c \).

\[
\Omega_0 = \frac{8\pi G\varrho_0}{3H_0^2}.
\]  

(21)

The subscript 0 has been attached to \( \Omega \) because the critical density \( \varrho_c \) changes with cosmic epoch, as does \( \Omega \). It is convenient to refer any cosmic density to \( \varrho_c \). For example, we will often refer to the density parameter of baryons, \( \Omega_B \), or of visible matter, \( \Omega_{\text{vis}} \), or of dark matter, \( \Omega_{\text{dark}} \), and so on – these are convenient ways of describing the relative importance of different contributions to \( \Omega_0 \).
Density Parameter in the Vacuum Fields

A density parameter associated with $\rho_v$ can now be introduced, in exactly the same way as the density parameter $\Omega_0$ was defined.

$$\Omega_\Lambda = \frac{8\pi G \rho_v}{3H_0^2} \quad \text{and so} \quad \Lambda = 3H_0^2\Omega_\Lambda.$$  \hspace{1cm} (22)

The dynamical equations (5) and (6) can now be written

$$\ddot{a} = -\frac{\Omega_0 H_0^2}{2a^2} + \Omega_\Lambda H_0^2 a; \hspace{1cm} (23)$$

$$a^2 = \frac{\Omega_0 H_0^2}{a} - \frac{c^2}{\mathcal{R}^2} + \Omega_\Lambda H_0^2 a^2. \hspace{1cm} (24)$$

A traditional way of rewriting these relations is in terms of a deceleration parameter $q_0$ defined by $q_0 = -\ddot{a}/\dot{a}^2$ at the present epoch. Then, in terms of $\Omega_0$ and $\Omega_\Lambda$, we find,

$$q_0 = \frac{\Omega_0}{2} - \Omega_\Lambda. \hspace{1cm} (25)$$
Density Parameters in Matter and Vacuum Fields

We can now substitute the values of $a$ and $\dot{a}$ at the present epoch, $a = 1$ and $\dot{a} = H_0$, into (23) to find the relation between the curvature of space, $\Omega_0$ and $\Omega_\Lambda$.

$$\frac{c^2}{\Re^2} = H_0^2[\left(\Omega_0 + \Omega_\Lambda\right) - 1],$$

(26)

or

$$\kappa = \frac{1}{\Re^2} = \frac{\left(\Omega_0 + \Omega_\Lambda\right) - 1}{\left(c^2 / H_0^2\right)}.$$ (27)

A common practice is to introduce a density parameter associated with the curvature of space at the present epoch $\Omega_K$ such that

$$\Omega_K = -\frac{c^2}{H_0^2\Re^2}$$

(28)

Then, equation (26) becomes

$$\Omega_0 + \Omega_\Lambda + \Omega_K = 1.$$ (29)
Density Parameters in Matter and Vacuum Fields

Thus, the condition that the spatial sections are flat Euclidean space becomes

$$(\Omega_0 + \Omega_\Lambda) = 1.$$ \hspace{1cm} (30)

The radius of curvature $R(t)$ of the spatial sections of these models change with scale factor as $R(t) = aR$ and so, if the space curvature is zero now, it must have been zero at all times in the past. This is one of the great attractions of the simplest inflationary picture of the early Universe.
Estimating the Value of $\Omega_\Lambda$

The theoretical value of $\Omega_\Lambda$ can be estimated using simple concepts from quantum field theory. Heisenberg’s Uncertainty Principle states that a virtual pair of particles of mass $m$ can exist for a time $t \sim \hbar/mc^2$, corresponding to a maximum separation $x \sim \hbar/mc$. Hence, the typical density of the vacuum fields is $\rho \sim m/x^3 \approx c^3m^4/\hbar^3$.

The mass density in the vacuum fields is unchanging with cosmic epoch and so, adopting the Planck mass for $m_{\text{Pl}} = (hc/G)^{1/2} = 5.4 \times 10^{-8} = 3 \times 10^{19}$ GeV, the mass density corresponds to about $10^{97}$ kg m$^{-3}$, about $10^{120}$ times greater than permissible values at the present epoch which correspond to $\rho_v \leq 10^{-27}$ kg m$^{-3}$.

This is quite a problem. If the inflationary picture of the very early Universe is taken seriously, this is exactly the type of force which drove the inflationary expansion. Then, we have to explain why $\rho_v$ decreased by a factor of about $10^{120}$ at the end of the inflationary era.
Confrontation with Observation

These are the models which are to be tested observationally. The revolution of the last 10 years has been that there is remarkable agreement between completely independent approaches to estimating basic cosmological parameters. These include:

- Supernovae of Type 1A
- Fluctuation spectrum and polarisation of the Cosmic Microwave Background Radiation
- The power spectrum of galaxies from the Sloan Digital Sky Survey and the AAO 2dF galaxy survey.
- Mass density of the Universe from the infall velocities of galaxies into large scale structures.
- The formation of the light elements.
- Cosmic time scale from the theory of stellar evolution and nucleocosmochronology.
- The value of Hubble’s constant from the HST Key Project.
Supernovae of Type 1a

The **Type 1a supernovae** are associated with the explosions of white dwarfs in binary systems. The explosion mechanism is probably nuclear deflagration associated with mass transfer from the companion onto the surface of the white dwarf which destroys the star. This process is expected to result in a uniform class of explosions.

Type 1a supernovae have dominated methods for extending the redshift-distance relation to large redshifts. They are very bright explosions and are observed to have remarkably standard properties, particularly when corrections are made for the luminosity-width relation.
Results of the combined ESSENCE and SNLP Projects

The ESSENCE project has the objective of measuring the redshifts and distances of about 200 supernovae. The Supernova Legacy Project aims to obtain distances for about 500 supernovae. The observations are consistent with a finite and positive value of the cosmological constant. Luminosity distance modulus vs. redshift for the ESSENCE, SNLS, and nearby SNe Ia (shown in red).

For comparison the overplotted solid line and residuals are for a $\Lambda$CDM model with $w = -1$, $\Omega_0 = 0.27$ and $\Omega_\Lambda = 0.73$. 
The Temperature Fluctuations and Polarisation of the Cosmic Microwave Background Radiation

These topics will be discussed in much more detail in the next lecture. The important point is the remarkable agreement between the predicted and observed power-spectrum of the temperature fluctuations which provide very powerful constraints on cosmological parameters, as will be summarised later.
Acoustic Peaks in the Galaxy Power-Spectra

The acoustic peaks in the distribution of galaxies at redshifts $z \leq 0.5$ correspond to the temperature maxima in the power-spectrum of fluctuations in the Cosmic Microwave Background Radiation imprinted at $z = 1000$. 
Redshift Distortions due to Infall into Large-scale Density Perturbations

The two-dimensional correlation function for galaxies selected from the 2dF Galaxy Redshift Survey. The flattening in the vertical direction is due to the infall of galaxies into large-scale density perturbations. The elongations along the central vertical axis are associated with the velocity dispersion in clusters of galaxies. The inferred overall density parameter is $\Omega_0 = 0.25$. 
Formation of the Light Elements by Primordial Nucleosynthesis

The predicted primordial abundances of the light elements as a function of the present baryon-to-photon ratio in the form \( \eta = 10^{10} n_B/n_\gamma = 274 \Omega_B h^2 \). \( Y_P \) is the abundance of helium by mass, whereas the abundances for D, \(^3\text{He}\) and \(^7\text{Li}\) are plotted as ratios by number relative to hydrogen. The widths of the bands reflect the theoretical uncertainties in the predictions.
Steigman finds a best fitting value $\Omega_B h^2 = 0.022^{+0.003}_{-0.002}$. This can be compared with the independent estimate from the power-spectrum of fluctuations of the Cosmic Microwave Background Radiation $\Omega_B h^2 = 0.0224 \pm 0.0009$. 

Helium

Deuterium

Lithium
Cosmic Time-scales

The Ages of the Oldest Globular Clusters:

Bolte (1997) :
\[ T_0 = 15 \pm 2.4 \text{ (stat)} \pm 4 \text{ (syst)} \text{ Gy} \]

Chaboyer (1998) :
\[ T_0 = (11.5 \pm 1.3) \text{ Gy} \]

Nucleocosmochronology

CS 22892-052 has iron abundance is 1000 times less than the solar value. A number of species never previously observed in such metal-poor stars were detected, as well a single line of thorium. A lower limit to the age of the star is
\[ (15.2 \pm 3.7) \times 10^9 \text{ years} . \]

Schramm found a lower limit to the age of the Galaxy of \( 9.6 \times 10^9 \) years and his best estimates of the age of the Galaxy are somewhat model-dependent, but typically ages of about \( (12 – 14) \times 10^9 \) years.
Hubble’s Constant

The controversies of the 1970s and 1980s have been resolved thanks to a very large effort by many observers to improve knowledge of the distances to nearby galaxies.

- Final result of HST key project $H_0 = 72 \pm 7 \, (1-\sigma) \, \text{km s}^{-1} \, \text{Mpc}^{-1}$.

- Other estimates:
  - Sandage and Tammann: Type Ia supernovae $H_0 = 59 \, \text{km s}^{-1} \, \text{Mpc}^{-1}$.
  - Gravitational time delays $H_0 = (60 - 65) \, \text{km s}^{-1} \, \text{Mpc}^{-1}$.
  - Sunyaev-Zeldovich effect $H_0 = (50 - 60) \, \text{km s}^{-1} \, \text{Mpc}^{-1}$. 
The Concordance Model

This set of parameters is consistent with all observations listed above. The errors are the quoted 1-$\sigma$ errors from the WMAP observations alone, assuming a flat world model, $\Omega_0 + \Omega_\Lambda = 1$.

- Curvature of Space $\Omega_\Lambda + \Omega_0 = 1; \kappa = 0$
- Hubble’s constant $H_0 = 73^{+3}_{-3}$ km s$^{-1}$ Mpc$^{-1}$
- Baryonic density parameter $\Omega_B h^2 = 0.0223^{+0.0007}_{-0.0009}$
- Cold Dark Matter density parameter $\Omega_D h^2 = 0.127^{+0.007}_{-0.013}$
- Total Matter density parameter $\Omega_0 h^2 = \Omega_B h^2 + \Omega_D h^2$
- Density Parameter in Vacuum Fields $\Omega_\Lambda = 1 - \Omega_0 h^2$
- Optical Depth for Thomson Scattering on Reheating $\tau = 0.09^{+0.03}_{-0.03}$
The Properties of the Concordance Model

Adopting $H_0 = 73$ km s$^{-1}$ Mpc$^{-1}$, we find the following self-consistent set of parameters:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hubble’s constant</td>
<td>$H_0 = 73$ km s$^{-1}$ Mpc$^{-1}$</td>
</tr>
<tr>
<td>Curvature of Space</td>
<td>$\Omega_{\Lambda} + \Omega_0 = 1$</td>
</tr>
<tr>
<td>Baryonic density parameter</td>
<td>$\Omega_B = 0.04$</td>
</tr>
<tr>
<td>Cold Dark Matter density parameter</td>
<td>$\Omega_D = 0.24$</td>
</tr>
<tr>
<td>Total Matter density parameter</td>
<td>$\Omega_0 = 0.28$</td>
</tr>
<tr>
<td>Density Parameter in Vacuum Fields</td>
<td>$\Omega_{\Lambda} = 0.72$</td>
</tr>
<tr>
<td>Optical Depth for Thomson Scattering on Reheating</td>
<td>$\tau = 0.09$</td>
</tr>
</tbody>
</table>

The remarkable feature of these figures is that they are now known to better than 10% accuracy. This is an extraordinary revolution. We live in the era of precision cosmology.

But there is also a huge challenge – we need to understand the physics to better than 10% to determine the cosmological parameters with improved precision.
The Properties of the Concordance Model

It therefore is sensible to regard this as the framework model for cosmological studies.

The Friedman equation is:

$$\dot{a}^2 = \frac{\Omega_0 H_0^2}{a} - \frac{c^2}{\gamma^2} + \Omega_\Lambda H_0^2 a^2. \quad (31)$$

Using the relation $a = 1/(1 + z)$, we find

$$\frac{dz}{dt} = -H_0(1 + z)[(1 + z)^2(\Omega_0 z + 1) - \Omega_\Lambda z(z + 2)]^{1/2}. \quad (32)$$

Cosmic time $t$ measured from the Big Bang follows immediately by integration

$$t = \int_0^t dt = -\frac{1}{H_0} \int_\infty^z \frac{dz}{\gamma^2 (1 + z)[(1 + z)^2(\Omega_0 z + 1) - \Omega_\Lambda z(z + 2)]^{1/2}}. \quad (33)$$
The Properties of the Concordance Model

The evidence suggests that we live in a Universe with zero spatial curvature, $R \rightarrow \infty$, and so $\Omega_0 + \Omega_\Lambda = 1$. This result simplifies the time-redshift relation:

$$t = \int_0^t dt = -{1 \over H_0} \int_0^\infty {dz \over (1 + z)\left[\Omega_0(1 + z)^3 + \Omega_\Lambda\right]^{1/2}}. \quad (34)$$

The cosmic time–redshift relation becomes

$$t = {2 \over 3H_0\Omega_\Lambda^{1/2}} \ln \left( {1 + \cos \theta \over \sin \theta} \right) \quad \text{where} \quad \tan \theta = \left( \frac{\Omega_0}{\Omega_\Lambda} \right)^{1/2} (1 + z)^{3/2}. \quad (35)$$

The present age of the Universe follows by setting $z = 0$

$$t_0 = {2 \over 3H_0\Omega_\Lambda^{1/2}} \ln \left[ \frac{1 + \Omega_\Lambda^{1/2}}{(1 - \Omega_\Lambda)^{1/2}} \right]. \quad (36)$$

If we adopt $\Omega_\Lambda = 0.72$ and $\Omega_0 = 0.28$, the age of the world model is

$$T_0 = 0.983H_0^{-1} = 1.32 \times 10^{10} \text{ years}. \quad (37)$$
The Properties of the Concordance Model

The expression for the comoving radial distance coordinate for $\Omega_0 + \Omega_\Lambda = 1$ is

$$r = \int_{t_1}^{t_0} c \frac{dt}{a} = -\frac{c}{H_0} \int_{\infty}^{z} dz \left[ \Omega_0 (1 + z)^3 + \Omega_\Lambda \right]^{1/2}.$$  \hspace{1cm} (38)